# CONTINUOUS VARIABLE TOMOGRAPHIC MEASUREMENTS

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ABSTRACT. Using a recent result of Albini *et al.* [1] to represent quantum homodyne tomography in terms of a single observable (as a normalized positive operator measure) we construct a generalized Markov kernel which transforms (the measurement outcome statistics of) this observable into (the measurement outcome statistics of) a covariant phase space observable. We also consider the inverse question. Finally, we add some remarks on the quantum theoretical justification of the experimental implementations of these observables in terms of balanced homodyne and 8-port detection techniques, respectively.

## 1. INTRODUCTION

There are two main approaches to continuous variable quantum tomography. The first one applies measurements of rotated quadratures $Q_\theta$, $\theta \in [0, 2\pi)$, the second one uses a measurement of the phase space observable $G_{|0\rangle}$ generated by the vacuum state $|0\rangle$. In both cases the state $\rho$ of the measured system can actually be reconstructed from the corresponding measurement statistics, either from the quadrature distributions $\rho^{Q_\theta}, \theta \in [0, 2\pi)$, or from the phase space (Husimi) distribution $\rho^{G_{|0\rangle}}$ [19, 17, 5, 15, 3, 16]

A direct comparison of the two tomographic methods has suffered from the fact that in the first case one needs to measure, in principle, infinitely many different, mutually noncommuting observables $Q_\theta, \theta \in [0, 2\pi)$, whereas in the latter approach a measurement of a single observable is sufficient. For each state $\rho$ the random sampling of the distributions $\rho^{Q_\theta}$, $\theta \in [0, 2\pi)$, can, however, easily be amalgamated into a probability bimeasure, which, when taken all together, define a unique informationally complete positive operator measure $E_{ht}$, the POM of the quantum homodyne tomography [1].

In this letter we construct a generalized Markov kernel which turns (the statistics of) the homodyne tomography observable $E_{ht}$ into (the statistics of) the phase space (Husimi) observable $G_{|0\rangle}$. We also demonstrate that there is no simple inverse construction. Finally, we briefly comment on the theoretical background of the physical implementation of the observables $E_{ht}$ and $G_{|0\rangle}$ via balanced homodyne detection and 8-port homodyne detection tehniques, respectively.

## 2. Notations and the two POMs

Let $h_n, n \in \mathbb{N}$, denote the (normalized) Hermite functions in $\mathcal{H} = L^2(\mathbb{R})$, $N$ the corresponding number operator, and let $Q$ and $P$ stand for the usual position and momentum operators, respectively. Let $\mathsf{Q}$ denote the spectral measure of $Q$ so that the quadrature observable $\mathsf{Q}_\theta$ is the spectral measure of the rotated position operator $e^{i\theta N} Q e^{-i\theta N}$. For any state $\rho$, positive trace one operator on $\mathcal{H}$, we let $\rho^{\mathsf{Q}_\theta}$ denote the density of the probability distribution $X \mapsto \mathrm{tr}\,[\rho\,\mathsf{Q}_\theta(X)]$, that is, $\mathrm{tr}\,[\rho\,\mathsf{Q}_\theta(X)] = \int_X \rho^{\mathsf{Q}_\theta}(x)\,dx$.

For each state $\rho$, the random sampling of the distributions $\rho^{\mathsf{Q}_\theta}$ can, indeed, be amalgamated into a probability bimeasure $\mathcal{B}([0, 2\pi)) \times \mathcal{B}(\mathbb{R}) \ni (\Theta, X) \mapsto \int_\Theta \mathrm{tr}\,[\rho\,\mathsf{Q}_\theta(X)]\frac{d\theta}{2\pi} \in [0,1]$, which can uniquely be extended to a probability measure on $\mathcal{B}([0, 2\pi) \times \mathbb{R})$ (we use the notation $\mathcal{B}(\cdots)$ for the Borel sets). As shown in [1], these probability measures define a unique normalized positive operator measure, the POM $\mathsf{E}_{\mathrm{ht}}$, with the property:[1]

$$(1) \quad \mathsf{E}_{\mathrm{ht}}(\Theta \times X) := \frac{1}{2\pi}\int_\Theta \mathsf{Q}_\theta(X)d\theta = \frac{1}{2\pi}\int_\Theta e^{i\theta N}\mathsf{Q}(X)e^{-i\theta N}d\theta, \quad \Theta \in \mathcal{B}([0, 2\pi)), \; X \in \mathcal{B}(\mathbb{R}).$$

For any state $\rho$, we let $\rho^{\mathsf{E}_{\mathrm{ht}}}$ denote the density of the probability measure $Z \mapsto \mathrm{tr}\,[\rho\mathsf{E}_{\mathrm{ht}}(Z)]$ with respect to $\frac{1}{2\pi}d\theta dx$, so that $\rho^{\mathsf{E}_{\mathrm{ht}}}(\theta, x) = \rho^{\mathsf{Q}_\theta}(x)$. Since the Radon transform of the Wigner function $\mathcal{W}_\rho$ of $\rho$ gives the quadrature distributions, that is $(\mathcal{R}\mathcal{W}_\rho)(\theta, x) = \rho^{\mathsf{Q}_\theta}(x)$, for almost all $\theta \in [0, 2\pi), x \in \mathbb{R}$, one thus has[2]

$$(2) \qquad \mathrm{tr}\,[\rho\mathsf{E}_{\mathrm{ht}}(\Theta \times X)] \quad = \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^\infty \rho^{\mathsf{E}_{\mathrm{ht}}}(\theta, x)d\theta dx = \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^\infty \rho^{\mathsf{Q}_\theta}(x)d\theta dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^\infty (\mathcal{R}\mathcal{W}_\rho)(\theta, x)d\theta dx.$$

Let $W_{qp} = e^{\frac{pq}{2}}e^{-iqP}e^{ipQ}$ denote the unitary Weyl operators. As well known, see e.g. [7, 21, 4, 11], any normalized POM $\mathsf{G}$ which satisfies the covariance condition with respect to the Weyl representation of the phase space translations, has an operator density $(q, p) \mapsto \frac{1}{2\pi}W_{qp}KW_{qp}^*$ defined by a unique positive trace one operator $K$, that is, for any state $\rho$,

$$(3) \qquad\qquad \mathrm{tr}\,[\rho\mathsf{G}(Z)] = \frac{1}{2\pi}\int_Z \mathrm{tr}\,[\rho W_{qp}KW_{qp}^*]\,dqdq.$$

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[1] Obviously, one could equally well define the POM $\widetilde{\mathsf{E}}_{\mathrm{ht}}(\Theta \times X) := \frac{1}{\pi}\int_\Theta \mathsf{Q}_\theta(X)d\theta$ for phase variable in $[0, \pi)$. Then $\mathsf{E}_{\mathrm{ht}}(\Theta \times X) = \frac{1}{2}\widetilde{\mathsf{E}}_{\mathrm{ht}}(\Theta \cap [0, \pi) \times X) + \frac{1}{2}\widetilde{\mathsf{E}}_{\mathrm{ht}}([(\Theta \cap [\pi, 2\pi)) - \pi] \times (-X))$ for all $\Theta \in \mathcal{B}([0, 2\pi))$ and $X \in \mathcal{B}(\mathbb{R})$. To have a direct use of some of the results of [1] we choose to work with $\mathsf{E}_{\mathrm{ht}}$, which, however, contains some redundancy in giving a double covering of $\mathbb{R}^2$.

[2] The third equality is obtained whenever $\rho$ is such that $\mathcal{W}_\rho \in L^1(\mathbb{R}^2)$, see, e.g. [1].

In particular, if $K = |h_0\rangle\langle h_0|$, then the phase space observable $\mathsf{G}_K$ generated by $K$ is denoted by $\mathsf{G}_{|0\rangle}$, and the density $\rho^{\mathsf{G}_{|0\rangle}}$ of the probability measure $Z \mapsto \mathrm{tr}\,[\rho\mathsf{G}_{|0\rangle}(Z)]$ is just the Husimi $Q$-function of the state $\rho$, that is, $\rho^{\mathsf{G}_{|0\rangle}}(q,p) = \frac{1}{2\pi}\langle W_{qp}h_0 \,|\, \rho W_{qp}h_0\rangle$, where $W_{qp}h_0 = |z\rangle$ is the coherent state, with $z = \frac{1}{\sqrt{2}}(q + ip)$.

## 3. Construction of the kernel

We shall construct a generalized Markov kernel $M_{|0\rangle}$ (defined on $\mathbb{R}^2 \times ([0, 2\pi) \times \mathbb{R})$) such that for any state $\rho$, the density $\rho^{\mathsf{G}_{|0\rangle}}$ is obtained from the density $\rho^{\mathsf{E}_{\mathrm{ht}}}$ as follows:

$$(4) \qquad \rho^{\mathsf{G}_{|0\rangle}}(q,p) = \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}} M_{|0\rangle}^{q,p}(\theta, x)\rho^{\mathsf{E}_{\mathrm{ht}}}(\theta, x)\,dx\,d\theta,$$

or, equivalently, for the POMs,

$$\mathsf{G}_{|0\rangle}(Z) = \int_0^{2\pi}\int_{\mathbb{R}} \underbrace{\left[\frac{1}{2\pi}\iint_Z M_{|0\rangle}^{q,p}(\theta, x)\,dq\,dp\right]}_{=:\,\mathcal{M}_{|0\rangle}(Z;\theta,x)} d\,\mathsf{E}_{\mathrm{ht}}(\theta, x) = \int_0^{2\pi}\int_{\mathbb{R}}\mathcal{M}_{|0\rangle}(Z;\theta, x)\,d\,\mathsf{E}_{\mathrm{ht}}(\theta, x).$$

where $Z \subseteq \mathbb{R}^2$ is compact (so that we can change the order of integration).

In a forthcoming paper [13] a general method is developed to construct such kernels for arbitrary phase space observables $\mathsf{G}_K$. This general method shows that the kernel can be obtained as a derivative of the Hilbert transform of the density of the generating operator $K$. In the special case of $K = |h_0\rangle\langle h_0|$ one may confirm this fact by a direct computation. We shall do so next.

Before doing that we note, however, that essentially the same computations have been carried out in a different context of quantum-state sampling where the matrix elements of the state, the density matrix, are constructed form the quadrature distributions in terms of a kernel function. In that connection these functions were called the sampling functions or the pattern functions, see, e.g. [15, Sect. 5] and the references therein.

The Hilbert transform of the function $h_0^2$, $h_0(x) = \frac{1}{\sqrt[4]{\pi}}e^{-\frac{1}{2}x^2}$, can easily be computed and one observes that it is the Dawson integral

$$(5) \qquad \mathrm{daw}(x) = e^{-x^2}\int_0^x e^{t^2}\,dt.$$

Following the method of [13] we now define the function

$$(6) \qquad M_{|0\rangle}^{q,p}(\theta, x) = 2\frac{\partial}{\partial x}\mathrm{daw}(x - q\cos\theta - p\sin\theta),$$

which is known [10] to be a bounded analytic function vanishing at infinity and having the values

$$(7) \qquad M_{|0\rangle}^{q,p}(\theta, x) = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{2^k (2k)!} H_{2k}(x - q\cos\theta - p\sin\theta).$$

It can easily be seen that $2\,d\mathrm{daw}(x)/dx$ is not a positive function (see for instance the picture in page 114 of [15]) so that $\mathcal{M}_{|0\rangle}(Z; \theta, x)$ is not a true conditional (or transition) probability; therefore we call $\mathcal{M}_{|0\rangle}(Z; \theta, x)$ a *generalized* Markov kernel.

We go on to show that (for compact sets $Z$)

$$(8) \qquad \mathsf{G}_{|0\rangle}(Z) = \int_0^{2\pi} \int_{\mathbb{R}} \left[ \frac{1}{2\pi} \int_Z M_{|0\rangle}^{q,p}(\theta, x)\,dq\,dp \right] d\mathsf{E}_{\mathrm{ht}}(\theta, x).$$

It suffices to show that the coherent state expectations of both sides of this equation are equal, that is, eq. (4) holds for the coherent states $\rho = |z\rangle\langle z|$. To verify this fact, we need some further notations.

For $(q, p)$ and $(u, v)$ in $\mathbb{R}^2$ we denote $z = (q + ip)/\sqrt{2}$, $w = (u + iv)/\sqrt{2}$, and

$$a := q\cos\theta + p\sin\theta = \sqrt{2}\,\mathrm{Re}(ze^{-i\theta}).$$

If $\tilde{w} = we^{-i\theta} = (\tilde{u} + i\tilde{v})/\sqrt{2}$ it follows that $\tilde{u} = u\cos\theta + v\sin\theta$ and

$$y := \tilde{u} - a = \sqrt{2}\,\mathrm{Re}\big((w - z)e^{-i\theta}\big).$$

Then

$$\langle w|\mathsf{G}_{|0\rangle}(Z)|w\rangle = \frac{1}{\pi} \int_Z \langle w|z\rangle\langle z|w\rangle d^2z = \frac{1}{2\pi} \int_Z e^{-|w-z|^2}\,dq\,dp,$$

$$\langle w|\mathsf{E}_{\mathrm{ht}}(\Theta \times X)|w\rangle = \frac{1}{2\pi} \int_\Theta \langle we^{-i\theta}|\mathsf{Q}(X)|we^{-i\theta}\rangle\,d\theta,$$

$$\langle \tilde{w}|\mathsf{Q}(X)|\tilde{w}\rangle = \frac{1}{\sqrt{\pi}} \int_X e^{-(x-\tilde{u})^2}\,dx.$$

It thus suffices to show that

$$e^{-|w-z|^2} = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} \int_{\mathbb{R}} M_{|0\rangle}^{q,p}(\theta, x)e^{-(x-\tilde{u})^2}\,\frac{d\theta\,dx}{2\pi}.$$

Now

$$\frac{1}{\sqrt{\pi}} \int_0^{2\pi} \int_{\mathbb{R}} M_{|0\rangle}^{q,p}(\theta, x) e^{-(x-\tilde{u})^2} \frac{d\theta dx}{2\pi}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \int_0^{2\pi} \frac{(-1)^k k!}{2^k (2k)!} \int_{\mathbb{R}} H_{2k}(x-a) e^{-(x-\tilde{u})^2} \frac{dx d\theta}{2\pi}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \int_0^{2\pi} \frac{(-1)^k k!}{2^k (2k)!} \underbrace{\int_{\mathbb{R}} H_{2k}(x) e^{-(x-y)^2} dx}_{= \sqrt{\pi} y^{2k} 2^{2k} \text{ (see [6, 7.374(6)])}} \frac{d\theta}{2\pi}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k k! 2^k}{(2k)!} \int_0^{2\pi} \underbrace{\left\{ \frac{1}{2^k} \sum_{l=0}^{2k} \binom{2k}{l} \left[(w-z)e^{-i\theta}\right]^l \left[\overline{(w-z)}e^{i\theta}\right]^{2k-l} \right\}}_{= y^{2k} \text{ since } y = \sqrt{2} \, \mathrm{Re}\left((w-z)e^{-i\theta}\right) = 2^{-1/2}\left((w-z)e^{-i\theta} + \overline{(w-z)}e^{i\theta}\right)} \frac{d\theta}{2\pi}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k)!} \sum_{l=0}^{2k} \binom{2k}{l} (w-z)^l \overline{(w-z)}^{2k-l} \underbrace{\int_0^{2\pi} e^{2(k-l)} \frac{d\theta}{2\pi}}_{= \delta_{l,k}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k)!} \cdot \frac{(2k)!}{k! \, k!} |w-z|^{2k} = \sum_{k=0}^{\infty} \frac{\left(-|w-z|^2\right)^k}{k!} = e^{-|w-z|^2},$$

showing that eq. (8) is, indeed, valid.

The construction of the Husimi POM $G_{|0\rangle}$ from the homodyne detection observable $E_{ht}$ by the generalized Markov kernel (7), as given in (8), is based on a direct computation. Due to the informational completeness of the phase space observable $G_{|0\rangle}$ one would expect that a similar reverse construction can also be given. In the appendix we demonstrate, however, that no simple inverse construction can be given.

## 4. Discussion

The measurements of the homodyne detection observable $E_{ht}$ and the phase space observable $G_{|0\rangle}$ constitute the basic measurements in the theory of (continous variable) quantum tomography. Due to their informational completeness their measurement outcome statistics $\rho^{E_{ht}}$ and $\rho^{G_{|0\rangle}}$ both separate states $\rho$. Together with the appropriate inversion formulas the state $\rho$ can, indeed, be deduced both from $\rho^{E_{ht}}$ as well as from $\rho^{G_{|0\rangle}}$. Therefore, the experimental implementations of measurements of $E_{ht}$ and $G_{|0\rangle}$ as well as the inversion formulas $\rho^{E_{ht}} \mapsto \rho$ and $\rho^{G_{|0\rangle}} \mapsto \rho$ are most important, and they have widely been analysed in the literature. In spite of that we find it appropriate to add a few comments on these well-known methods and on the relevance of the kernel constructed above.

The balanced homodyne detection is well-known and much used techique to measure a quandarture observable $Q_\theta$, where $\theta$ is the phase of the local oscillator, in coherent state $|z\rangle$, $z = |z|e^{i\theta}$, and $|z| \gg 1$. A rigorous quantum mechanical proof that the balanced homodyne detection observable tends, in the high amplitude limit $|z| \to \infty$, to the quadrature observable $Q_\theta$ has recently been worked out in [8], but see also [18] and [20]. The informational completeness of the quadrature observables $Q_\theta, \theta \in [0, \pi]$, is equally well-known; recent explicite proofs are given, for instance, in [3, 10]. The statistical sampling of various $Q_\theta$ is, again, well-known, see e.g. [15, 16], though the combination of them under a single observable $E_{ht}$ is a recent result [1]. To our knowledge the only known measurement implementation of $E_{ht}$, however, is the high amplitude balanced homodyne detection together with a random change of the phase $\theta$.

Phase space observables $G_K$, and, in particular, the Husimi POM $G_{|0\rangle}$, are other important examples of informationally complete observables. Indeed, $G_K$ is informationally complete if and only if the generating operator $K$ is such that $\mathrm{tr}\,[W_{qp}K] \neq 0$ for (almost all) $(q, p) \in \mathbb{R}^2$. This results is due to [2], see [12] for some further technical elaborations. The Arthurs-Kelly model as well as the 8-port homodyne detection scheme are, again, well-known measurement models for phase space observables $G_K$, the latter one being also experimentally feasible. In fact, there is a recent rigorous proof that any phase space observable $G_K$ can be obtained (as a high-amplitude limit of the local oscillator) of the 8-port homodyne detection observable, with the generating operator $K$ defined by the state of the (single-mode) parametric field [9].

The generalized Markov kernel (6) allows one to obtain the Husimi distribution $\rho^{G_{|0\rangle}}$ of any state $\rho$ from the homodyne detection distribution $\rho^{E_{ht}}$ by the formula (4), that is, the phase space observable $G_{|0\rangle}$ from the homodyne detection observable $E_{ht}$ by (8). This means, in particular, that any measurement of $G_{|0\rangle}$ can, in principle, be reduced to a measurement of $E_{ht}$.

APPENDIX: THE REVERSE PROBLEM

To solve the inverse problem, we try to find a function $N_{|0\rangle}^{\theta,x}(u, v)$ such that

$$(9) \qquad E_{ht}(\Theta \times X) = \iint_{\mathbb{R}^2} \left[ \int_\Theta \int_X N_{|0\rangle}^{\theta,x}(u, v) d\theta dx \right] \underbrace{dG_{|0\rangle|}(w)}_{=|w\rangle\langle w|dqdp/(2\pi)}$$

where $w = (u+iv)/\sqrt{2}$, and where the order of integration can be changed (at least for compact sets $X \subset \mathbb{R}$).

As before, we denote $z = (q+ip)/\sqrt{2}$, and $\tilde{z} = ze^{-i\theta} = (\tilde{q}+i\tilde{p})/\sqrt{2}$ so that $\tilde{q} = q\cos\theta + p\sin\theta$, $\tilde{p} = -q\sin\theta + p\cos\theta$, $q = \tilde{q}\cos\theta - \tilde{p}\sin\theta$, $p = \tilde{q}\sin\theta + \tilde{p}\cos\theta$, and $d\tilde{q}d\tilde{p} = dqdp$.

Operate on both sides of (9) by $\langle -z| \cdots |z\rangle$. Now

$$\langle -z|\mathsf{E}_{\mathrm{ht}}(\Theta \times X)|z\rangle = \frac{1}{2\pi}\int_{\Theta} \langle -e^{-i\theta}z|\mathsf{Q}(X)|e^{-i\theta}z\rangle d\theta$$

and

$$\langle -e^{-i\theta}z|\mathsf{Q}_0(X)|e^{-i\theta}z\rangle = \langle -\tilde{z}|\mathsf{Q}_0(X)|\tilde{z}\rangle = \frac{1}{\sqrt{\pi}}\int_X e^{i2\tilde{p}x-(x-\tilde{q})^2/2-(x+\tilde{q})^2/2}dx = \frac{1}{\sqrt{\pi}}\int_X e^{i2\tilde{p}x-x^2-\tilde{q}^2}dx.$$

On the other hand,

$$\langle -z|\mathsf{G}_{|0\rangle}(Z)|z\rangle = \frac{1}{2\pi}\iint_Z \langle -z|w\rangle\langle w|z\rangle dudv = \frac{1}{2\pi}\iint_Z e^{-|z|^2-|w|^2+2i\mathrm{Im}(z\overline{w})}dudv.$$

Thus, one must solve

$$\frac{1}{\sqrt{\pi}}e^{i2\tilde{p}x-x^2-\tilde{q}^2} = \iint_{\mathbb{R}^2} N^{\theta,x}_{|0\rangle}(u,v)e^{-|z|^2-|w|^2+2i\mathrm{Im}(z\overline{w})}dudv$$

or, equivalently, (note that $2i\mathrm{Im}(z\overline{w}) = i(pu - qv)$)

$$\frac{1}{\sqrt{\pi}}e^{i2\tilde{p}x-x^2-\tilde{q}^2+|z|^2} = \iint_{\mathbb{R}^2} N^{\theta,x}_{|0\rangle}(u,v)e^{-|w|^2}\cdot e^{ipu}e^{-iqv}dudv.$$

If one makes assumptions that the right hand side of the above equation (a function of $q$ and $p$) and the function $(u,v)\mapsto N^{\theta,x}_{|0\rangle}(u,v)e^{-(u^2+v^2)/2}$ are integrable, one can take take the double Fourier transform: one gets

$$N^{\theta,x}_{|0\rangle}(u,v)e^{-(u^2+v^2)/2} = \frac{1}{4\pi^2}\frac{1}{\sqrt{\pi}}\iint_{\mathbb{R}^2} e^{i2\tilde{p}x-x^2-\tilde{q}^2+|z|^2}\cdot e^{-ipu}e^{iqv}dqdp.$$

If the last integral would exist, then one could change integration variables to get

$$\iint_{\mathbb{R}^2} e^{i2\tilde{p}x-x^2-\tilde{q}^2+|z|^2}\cdot e^{-ipu}e^{iqv}dqdp = \iint_{\mathbb{R}^2} e^{i2\tilde{p}x-x^2-\tilde{q}^2+|\tilde{z}|^2-i(\tilde{q}\sin\theta+\tilde{p}\cos\theta)u+i(\tilde{q}\cos\theta-\tilde{p}\sin\theta)v}d\tilde{q}d\tilde{p}$$

where the argument of the exponential function of the integrand is

$$i2\tilde{p}x - x^2 - \tilde{q}^2 + |\tilde{z}|^2 - i(\tilde{q}\sin\theta + \tilde{p}\cos\theta)u + i(\tilde{q}\cos\theta - \tilde{p}\sin\theta)v$$

$$= -\frac{1}{2}\tilde{q}^2 + \frac{1}{2}\tilde{p}^2 + i\tilde{q}(-u\sin\theta + v\cos\theta) + i\tilde{p}(2x - u\cos\theta - v\sin\theta) - x^2.$$

Certainly the above integral does not exist even as a tempered distribution due to the $\frac{1}{2}\tilde{p}^2$-term; for example, if $u = v = x = 0$, then one would have

$$N^{\theta,0}_{|0\rangle}(0,0) = \frac{1}{4\pi^2}\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{\frac{1}{2}\tilde{p}^2}d\tilde{p}\int_{\mathbb{R}} e^{-\frac{1}{2}\tilde{q}^2}d\tilde{q}$$

which clearly does not exist. To conlude, there is no function $N^{\theta,x}_{|0\rangle}(u,v)$ with the above properties which would allow one to write $\mathsf{E}_{\mathrm{ht}}(\Theta \times X) = \iint_{\mathbb{R}^2}\left[\int_{\Theta}\int_X N^{\theta,x}_{|0\rangle}(u,v)d\theta dx\right]d\mathsf{G}_{|0\rangle|}(w).$

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